Ultrametric Gelfand Transforms

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Abstract

Let $\mathbb{K}$ be an algebraically closed field, complete for a non-trivial ultrametric absolute value, and let A be a commutative normed $\mathbb{K}$-algebra with identity. We call *multiplicative spectrum* of A the set $Mult(A, \|\cdot\|)$ of continuous multiplicative semi-norms on A. We denote by $Mult(\mathbb{K}[X])$ the set of multiplicative semi-norms on the polynomial algebra $\mathbb{K}[X]$. Both sets of semi-norms are endowed with the topology of pointwise convergence. We also denote by $\mathcal{X}(A, \mathbb{K})$ the set of $\mathbb{K}$-algebra homomorphism from A onto $\mathbb{K}$. Unlike in complex analysis, there might exist some maximal ideals in A which are not the kernel of elements of $\mathcal{X}(A, \mathbb{K})$. In A, we can define two kinds of Gelfand transform. $\mathcal{G}_A$ and $\mathcal{GM}_A$. The first one, denoted by $\mathcal{G}_A$ is similar to that in complex analysis, consisting of associating to each element f of A the mapping $\widehat{f}$ from $\mathcal{X}(A, K)$ to K defined as $\widehat{f}(\chi) = \chi(f)$, $(\chi \in \mathcal{X}(A, K))$. The second, denoted by $\mathcal{GM}_A$ consists of associating to each element f of A the mapping f^* from $Mult(A, \| \,.\, \|)$ to $Mult(K[x])$ defined as $f^*(\phi)(P) = \phi(P \circ f)$. This transform allows us to interpret any element of A as a continuous function defined on a compact space $(Mult(A, \|\cdot\|))$ and with value in a locally compact space $(Mult(\mathbb{K}[X]))$. We study these transforms and particularly the injectivity of the second. We deduce some spectral properties of this injectivity. Given $\phi \in Mult(A, \| \,.\, \|)$, we will denote by Z_ϕ the mapping from A into $Mult(K[x])$ defined as $Z_\phi(f) = f^*(\phi)$. We show that each function Z_ϕ is continuous. Moreover we put a metric topology δ on $(Mult(\mathbb{K}[X]))$ such that the family of functions Z_ϕ, $\phi \in Mult(A, \| \,.\, \|)$ is equicontinuous.